

Hilbert von Neumann Modules

versus

Concrete von Neumann Modules

Michael Skeide, May 2012

Abstract

Hilbert von Neumann modules and concrete von Neumann modules are the same thing.

Let G and H be Hilbert spaces. For $E \subset \mathcal{B}(G, H)$ denote by $[E]$ the strongly closed subspace of $\mathcal{B}(G, H)$ generated by E .

Bikram, Mukherjee, Srinivasan, and Sunder [BMSS12] say on Page 50, E is a **von Neumann corner** if $E = [E] \supset EE^*E$. It is **nondegenerate** if $\overline{\text{span}} EG = H$, $\overline{\text{span}} E^*H = G$. They say in [BMSS12, Definition 1.2(1)], a **Hilbert von Neumann module** over a von Neumann algebra \mathcal{A} is a von Neumann corner E with a normal isomorphism π from \mathcal{A} onto $[E^*E]$.

Let \mathcal{B} be a von Neumann algebra acting nondegenerately on the Hilbert space G . Skeide [Ske06, Definition 2] says, E is a **concrete von Neumann \mathcal{B} -module** if E is strongly closed, if it is a (right) \mathcal{B} -submodule^[1] of $\mathcal{B}(G, H)$ (that is, $E + E \subset E$ and $E\mathcal{B} \subset E$), if $E^*E \subset \mathcal{B}$ and if $\overline{\text{span}} EG = H$.

Proposition. *Let G and H be Hilbert spaces. For the subset E of $\mathcal{B}(G, H)$ denote by \mathcal{B} the strong closure in $\mathcal{B}(G)$ of the algebra generated by E^*E . Then the following are equivalent:*

1. E is a nondegenerate von Neumann corner.
2. E is a Hilbert von Neumann module over \mathcal{B} with $\pi = \text{id}_{\mathcal{B}}$ satisfying $\overline{\text{span}} EG = H$.
3. E is a concrete von Neumann \mathcal{B} -module.

Moreover, if E is a Hilbert von Neumann module over \mathcal{A} satisfying $\overline{\text{span}} EG = H$, then E is a concrete von Neumann $\pi(\mathcal{A})$ -module.

PROOF. This is immediate from the definitions. ■

Observation. Similar statements, which we omit phrasing here, are true for **Hilbert von Neumann bimodules** ([BMSS12, Definition 1.2(3)]) and **concrete von Neumann correspondences** ([Ske06, Definition 3]).

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^[1] In [Ske06] we omitted to repeat that a module is closed under addition.

Remark. *Von Neumann modules* have been introduced in Skeide [Ske00, Definition 4.4] as (pre-) Hilbert modules over von Neumann algebras for which the linking algebra is a von Neumann algebra (cf. [BMSS12, Proposition 1.1]). Immediately after ([Ske00, Proposition 4.5]), it is shown that this is equivalent, to that E is a concrete von Neumann module (of course, not calling it a concrete von Neumann module, as this definition is made not before [Ske06]). The main work is to transform the abstract (pre-)Hilbert module E over the (concrete!) von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ into a concrete operator module $E \subset \mathcal{B}(G, H)$. However, such a procedure is known, and it is known, too, that the result is unique up to suitable unitary equivalence; see (for instance) Rieffel [Rie74, Proposition 6.10] or Murphy [Mur97, Section 3]. All the work for deriving results (for instance, in Skeide [Ske00, Ske01]) is done with concrete von Neumann modules (respectively, Hilbert von Neumann modules). This includes, in particular, self-duality ([Ske00, Theorem 4.16], using cyclic decomposition [Ske00, Proposition 3.8], polar decomposition [Ske00, Proposition 2.10], and quasi orthonormal bases [Ske00, Theorem 4.11]; cf. [BMSS12, Proposition 1.9]). And it includes the tensor product of von Neumann correspondences ([Ske01, Equation (4.2.2) and Proposition 4.2.24], cf. [BMSS12, Section 3]). Just, a formal definition that frees us from the “burden” to transform an abstract (pre-)Hilbert module over a (concrete) von Neumann algebra into a concrete one, has not been given before [Ske06].

References

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